INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Second Year, First Semester, 2023-24
Statistics - II, Final Examination, November 20, 2023
Maximum Marks: 50 **Time: 3 hours**

1. $X_1, X_2, \ldots, X_n$ is a random sample from $N(0, \sigma^2)$ where $\sigma^2 > 0$ is unknown. Consider testing $H_0 : \sigma^2 \leq 1$ versus $H_1 : \sigma^2 > 1$ at the significance level $0 < \alpha < 1$.
(a) Show that UMP test exists.
(b) Consider the test which rejects H_0 when $\sum_{i=1}^{n}(X_i - \bar{X})^2 > \chi^2_{n-1}(1-\alpha)$. Show that this test has level α.
(c) Is the test given in (b) the UMP test? Justify. [2+3+5]

2. $X_1, X_2, \ldots, X_n$ is a random sample from $N(\mu, 1)$ where $-\infty < \mu < \infty$ is unknown.
(a) Derive the generalized likelihood ratio test (GLRT) for testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ at the significance level α for any fixed μ_0.
(b) Using the test obtained in (a) construct a $100(1-\alpha)\%$ confidence interval for μ. [7+3]

3. Suppose that X, the weekly number of accidents at a street junction, has the Poisson distribution with parameter $\lambda > 0$. Let $X_1, \ldots, X_5$ be a random sample with the observed counts of $1, 0, 3, 2, 2$. Assume that the prior distribution on λ is Exponential with mean 1.
(a) Find the posterior distribution of λ. What are the posterior mean and standard deviation?
(b) How does one construct the $100(1-\alpha)\%$ HPD credible interval for λ?
(c) Find the posterior odds ratio and the Bayes factor for testing $H_0 : \lambda \leq 2$ versus $H_0 : \lambda > 2$. (Numerical answers are not needed.) [7+3+4]

4. Consider a random sample $X_1, X_2, \ldots, X_n$ from $N(\theta, \theta^2)$ where $\theta > 0$.
(a) What is the minimal sufficient statistics for θ?
Consider the statistics $T_1(X_1, \ldots, X_n) = \bar{X} = \sum_{i=1}^{n} X_i/n$ and
$T_2(X_1, \ldots, X_n) = \sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2/n}$.
(b) Are these unbiased estimates of θ? Are they consistent?
(c) Find the asymptotic distributions of T_1 and T_2.
(d) As an estimator of θ, what is the asymptotic relative efficiency of T_1 with respect to T_2? [3+3+8+2]